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Free convection at an axisymmetric stagnation point

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Free-convective flow in the neighbourhood of the upper pole of a heated sphere at high Grashof number is considered. Direct buoyancy effects have been studied previously, and it is known that the unsteady boundary-layer solution may terminate in singular behaviour. For a spatially varying surface temperature, a self-induced pressure gradient is present. The effects of this are examined in detail, and it is shown that a singular behaviour may again terminate the solution. The competing effects of buoyancy and induced pressure gradient are examined to delineate those cases in which a steady-state flow is achieved.

1. Introduction

In this paper we are concerned with what might be described as mixed free convection in the neighbourhood of the upper pole of a heated sphere. The classical situation of free convection from a uniformly heated sphere, in which the fluid motion arises from the buoyancy forces, has received considerable attention in recent years. Potter & Riley (1980) considered the steady flow situation. They calculated the boundary-layer flow over the entire sphere and outlined the singular behaviour of the converging flow at the upper pole as the fluid erupts into a buoyant plume. An important feature of their work is the interpretation of experiment that it afforded, in both the boundary layer on the sphere and the plume above it. In a later paper, Brown & Simpson (1982) elaborated on the structure of the singular behaviour at the upper pole. They also considered the unsteady case of an impulsively heated sphere. From a local analysis at the upper pole they showed, by both numerical and analytic means, how the thickening boundary layer suddenly erupts to initiate the plume. The eruption manifests itself as a singularity in the solution of the boundary-layer equations. Subsequently, Awang & Riley (1983) confirmed the predicted behaviour at the upper pole by calculating the boundary-layer flow over the whole sphere.

Mixed convection is usually associated with the interaction between forced convection and buoyancy-driven free convection, acting to either reinforce or oppose one another, as for example in the early definitive investigation of Merkin (1969). In a recent paper, which contains a comprehensive list of references on mixed convection, Daniels (1992) considers mixed convection of a slightly different type. A thermally stratified fluid flows past an insulated semi-infinite horizontal flat plate. The thermal stratification results in a horizontal pressure gradient, and when this opposes the motion a singular breakdown of the solution of the boundary-layer equations occurs at a finite distance from the leading edge of the plate. In an earlier paper, Amin & Riley (1990) showed how temperature variations along a horizontal plane boundary would result in a self-induced horizontal pressure gradient. In particular they showed that if a temperature distribution that falls away quadratically in space from a given point is suddenly imposed on the boundary, the induced pressure gradient drives the fluid towards that point and eruption of fluid results, made manifest by a finite-time singularity in the boundary-layer solution. In a later paper, Amin & Riley (1995), the authors embedded this within a stagnation-point flow and showed, in particular, how in this mixed convective situation the singular behaviour could be suppressed.

At the beginning of this section we alluded to mixed free convection and it is, of course, the interaction between buoyancy-driven free convection and convection due to a self-induced pressure gradient with which we are concerned. We consider first the latter in isolation. We show how the sudden imposition of a suitably varying temperature distribution can result in a pressure gradient that forces the flow to converge on the stagnation point at the upper pole of a sphere. Again the solution of the boundary-layer equations terminates in a singularity which heralds the onset of an eruption of fluid from the surface. We analyse this singular behaviour, which differs in detail from those previously studied. We then demonstrate that the introduction of direct buoyancy can suppress this singular, eruptive behaviour. Similarly we show how the buoyancy-driven singular behaviour described by Brown & Simpson (1982) may be suppressed by the presence of a suitable induced pressure gradient.

2. Governing equations

For unsteady, free-convective flow over a heated sphere the equations of motion are, for a Boussinesq fluid and in the high-Grashof number boundary-layer limit,

$$\frac{\partial}{\partial x}(u\sin x)+\frac{\partial}{\partial y}(v\sin x)=0, \tag{2.1a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}-Gr^{1/5}\theta\sin x, \tag{2.1b}$$

$$0=-\frac{\partial p}{\partial y}+\theta\cos x, \tag{2.1c}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{\sigma}\frac{\partial^2 \theta}{\partial y^2}. \tag{2.1d}$$

In these equations the Grashof number is defined as

$$Gr=\frac{ga^3\beta(T_0-T_\infty)}{\nu^2}, \tag{2.2}$$

where $\boldsymbol{g}=(-g\cos x,\ g\sin x,\ 0)$ is the gravity vector, β is the coefficient of thermal expansion such that $(\rho-\rho_\infty)/\rho_\infty=\beta(T-T_\infty)$, with ρ density, T temperature and a subscript ∞ denoting conditions in the ambient fluid, a is the sphere radius, ν the kinematic viscosity, and T_0 is a reference temperature. With x measured from the upper pole of the sphere, ax measures distance along the surface and $aGr^{-1/5}y$ normal to it. With $U_0=\nu Gr^{2/5}/a$ as a typical velocity, the velocity components parallel and perpendicular to the surface are $U_0 u$ and $Gr^{-1/5}U_0 v$ respectively; the pressure is $\rho_\infty U_0^2 p$ and time at/U_0. Finally, θ is defined as $\theta=(T-T_\infty)/(T_0-T_\infty)$, and σ is the Prandtl number.

The boundary conditions for equations (2.1) require, if the flow is initiated at $t = 0$ by changing the temperature of the boundary from its ambient value,

$$\left.\begin{aligned} &u = v = \theta = 0 \quad \text{for} \quad y > 0, \quad t = 0, \\ &u, \theta, p \to 0 \quad \text{as} \quad y \to \infty, \quad t > 0, \\ &u = v = 0 \quad \text{for} \quad y = 0, \quad t > 0, \end{aligned}\right\} \tag{2.3}$$

together with a condition on θ, say θ_w, at $y = 0$ for $t > 0$. This we take as

$$\theta_w = Gr^{-1/5} b_0 + b_1 x^2, \tag{2.4}$$

where b_0, b_1 are constants. The disparate order of magnitude between the two terms of (2.4) implies, as we shall see, comparability between direct buoyancy effects, and indirect effects in the form of a self-induced pressure gradient parallel to the boundary. Following Brown & Simpson (1982) we develop the solution, close to the upper pole of the sphere, by writing

$$u(x,y,t) = x|b_1|^{2/5} u_0(\tilde{y}, \tilde{t}) + O(x^3), \tag{2.5a}$$

$$v(x,y,t) = |b_1|^{1/5} v_0(\tilde{y}, \tilde{t}) + O(x^2), \tag{2.5b}$$

$$p(x,y,t) = Gr^{-1/5}|b_1|^{3/5} p_0(\tilde{y}, \tilde{t}) + x^2 |b_1|^{4/5} p_1(\tilde{y}, \tilde{t}) + O(x^4), \tag{2.5c}$$

$$\theta(x,y,t) = Gr^{-1/5}|b_1|^{4/5} \theta_0(\tilde{y}, \tilde{t}) + x^2 |b_1| \theta_1(\tilde{y}, \tilde{t}) + O(x^4), \tag{2.5d}$$

where

$$\tilde{y} = |b_1|^{1/5} y, \quad \tilde{t} = |b_1|^{2/5} t.$$

Substituting (2.5) into (2.1), and noting $Gr \gg 1$, gives the following set of equations for the unknown coefficients,

$$2u_0 + \frac{\partial v_0}{\partial \tilde{y}} = 0, \tag{2.6a}$$

$$\frac{\partial u_0}{\partial \tilde{t}} + u_0^2 + v_0 \frac{\partial u_0}{\partial \tilde{y}} = -2p_1 + \frac{\partial^2 u_0}{\partial \tilde{y}^2} - \theta_0, \tag{2.6b}$$

$$0 = -\frac{\partial p_1}{\partial \tilde{y}} + \theta_1, \quad 0 = -\frac{\partial p_0}{\partial \tilde{y}} + \theta_0, \tag{2.6c, d}$$

$$\frac{\partial \theta_1}{\partial \tilde{t}} + 2u_0 \theta_1 + v_0 \frac{\partial \theta_1}{\partial \tilde{y}} = \frac{1}{\sigma} \frac{\partial^2 \theta_1}{\partial \tilde{y}^2}, \tag{2.6e}$$

$$\frac{\partial \theta_0}{\partial \tilde{t}} + v_0 \frac{\partial \theta_0}{\partial \tilde{y}} = \frac{1}{\sigma} \frac{\partial^2 \theta_0}{\partial \tilde{y}^2}, \tag{2.6f}$$

together with

$$\left.\begin{aligned} &u_0 = v_0 = \theta_i = 0, \quad \tilde{y} > 0, \quad \tilde{t} = 0 \quad (i = 0, 1), \\ &u_i, \theta_i, p_i \to 0, \quad \tilde{y} \to \infty, \quad \tilde{t} > 0 \quad (i = 0, 1), \\ &u_0 = v_0 = 0, \quad \theta_0 = \lambda, \quad \theta_1 = \operatorname{sgn} b_1, \quad \tilde{y} = 0, \quad \tilde{t} > 0, \end{aligned}\right\} \tag{2.7}$$

where $\lambda = |b_1|^{-4/5} b_0$ is the single parameter that characterizes the flow.

3. Solution procedure and results

We note first that if $b_1 \equiv 0$ the problem is that of a uniformly heated sphere as considered by Brown & Simpson (1982). The scalings in (2.5) are not, of course, appropriate in that case, but the equations studied by Brown & Simpson are recovered by setting $p_1 = \theta_1 \equiv 0$ in (2.6). For a cooled surface, that is with $b_0 < 0$, fluid flows away from the stagnation point at the upper pole and the solution evolves to a steady state. However for $b_0 > 0$ the flow converges onto the stagnation point, and erupts from it at a finite time. Such an eruption corresponds to a failure of the boundary-layer approximation, with the solution developing a singularity at a finite time. For the case $b_0 \equiv 0$ the flow is entirely due to the self-induced pressure gradient. From the hydrostatic balance (2.6*c*) we have

$$p_1 = -\int_{\tilde{y}}^{\infty} \theta_1 \,\mathrm{d}\tilde{y}, \tag{3.1}$$

and if we make the reasonable assumption that $\operatorname{sgn}\theta_1 = \operatorname{sgn} b_1$ we see that if the temperature increases away from the stagnation point the pressure gradient is favourable and a steady-state flow will result. However for $b_1 < 0$ the induced pressure gradient will force fluid towards the stagnation point, and such an accumulation of fluid will again result in an eruption, made manifest by the appearance of a singularity in the boundary-layer solution at a finite time. It is clear from the above discussion that if b_0, b_1 are both non-zero, then whether the solution of (2.6), (2.7) evolves to a steady state, or terminates in an eruptive singularity at a finite time, depends upon the delicate balance that exists between direct buoyancy and induced pressure gradient effects. Before considering that balance we examine in detail the flow due to the self-induced pressure gradient when $b_0 \equiv 0$.

The situation for $b_1 > 0$ is not, of itself, of great interest; the unsteady solution evolves to the solution of the steady-state equations. For $b_1 < 0$ our solution procedure is as follows. To accommodate the initial development of the boundary layer we introduce the new independent variable $\eta = \tilde{y}(1+\tilde{t})^{1/2}/\tilde{t}^{1/2}$. When this is introduced into (2.6), we set $\tilde{t} = 0$ and (2.6*e*) reduces to an ordinary differential equation which yields the pure conduction solution for θ_1, with all other variables zero, as the initial solution. From this initial state the solution is advanced in time by solving (2.6*a, b, c, e*) sequentially and iteratively as follows. All derivatives are represented by central differences, so that the numerical method is essentially of Crank–Nicolson type. At each new time step initial estimates for all variables are obtained, either by extrapolation from the two previous time levels or, at the first time step, by using the initial solution. From (2.6*c*) $p_1(\eta, \tilde{t})$ is first up-dated, this update of p_1 is then used in (2.6*b*) and the nonlinear equation for u_0 is solved iteratively; new estimates of v_0 and θ_1 follow from (2.6*a*) and (2.6*e*) respectively. This sequence of events is followed until the changes in all four dependent variables, following a global sweep through the equations, fall below some prescribed value, and we may advance a further step in time. As with other eruptive boundary-layer situations, in particular the unsteady free-convective problem considered by Brown & Simpson (1982), the normal velocity at the edge of the boundary layer, $v_{0\infty}$, $\max|u_0|$ and the boundary-layer thickness all increase indefinitely as the singular point $\tilde{t} = \tilde{t}_s$ is approached, as also do $\max|\theta_1|$ and $\max|p_1|$. The rapid increase in these quantities is becoming apparent when $\tilde{t}$ exceeds about 1.75. In all the calculations we report we have taken the edge of the computational domain at $\eta = \eta_\infty = 220$, with a step length $\delta\eta = 0.01$. The step length in time was initially $\delta\tilde{t} = 0.005$ but for $\tilde{t} > 1.6$ this was reduced, to maintain accuracy and to enable as close

$\tilde{t}$	u_m	$10^2\, u_m^{-1}$	θ_m	$10^2\, \theta_m^{-8/9}$	$v_{0\infty}$	$10^2\, v_{0\infty}^{-8/15}$
1.7460	42.30823	2.364	34.59391	4.286	1717.32334	1.882
1.7500	58.35377	1.714	49.65310	3.108	3018.10487	1.394
1.7540	93.27614	1.072	84.25230	1.943	6923.86535	0.895
1.7580	227.05044	0.440	229.86908	0.796	33977.76529	0.383

TABLE 1. Variation of $u_m = \max|u_0|$, $\theta_m = \max|\theta_1|$, $v_{0\infty}$ with $\tilde{t}$

an estimate of $\tilde{t}_s$ to be made as possible, to $\delta\tilde{t} = 0.0001$. The calculations were continued up to $\tilde{t} = 1.7582$ beyond which the mesh size and value of η_∞ were judged to be inadequate. A careful examination of the results shows that $\tilde{t}_s \approx 1.761$. Furthermore, as $\tau = \tilde{t}_s - \tilde{t} \to 0$ the quantities $v_{0\infty}\tau^{15/8}$, $\max(|u_0|\tau)$, $\max(|\theta_1|\tau^{9/8})$, $p_1(0, \tilde{t})\tau^2$ are approaching finite limits, as is $\eta_{max}\tau^{7/8}$ where η_{max} represents either the location of $\max|u_0|$ or $\max|\theta_1|$. These results are typical of other eruptive situations, but differ in detail. A sample of these results is set out in Table 1, and we remark that unit Prandtl number has been taken throughout. The behaviour of these quantities is consistent with the conjectured limiting forms above, as $\tau \to 0$.

To analyse this singular behaviour, as $\tau \to 0$, in more detail we write

$$\tilde{\eta} = \tau^{7/8}\tilde{y}, \quad u_0 = \tau^{-1}\tilde{u}_0, \quad v_0 = \tau^{-15/8}\tilde{v}_0, \quad \theta_1 = \tau^{-9/8}\tilde{\theta}_1, \quad p_1 = \tau^{-2}\tilde{p}_1.$$

When these variables are introduced into (2.6) we have, retaining only leading-order terms, and with $\tilde{\theta}_1$ eliminated

$$2\tilde{u}_0 + \frac{\partial \tilde{v}_0}{\partial \tilde{\eta}} = 0, \quad \tilde{u}_0 - \tfrac{7}{8}\tilde{\eta}\frac{\partial \tilde{u}_0}{\partial \tilde{\eta}} + \tilde{u}_0^2 + \tilde{v}_0\frac{\partial \tilde{u}_0}{\partial \tilde{\eta}} = -2\tilde{p}_1, \tag{3.2a, b}$$

$$\tfrac{9}{8}\frac{\partial \tilde{p}_1}{\partial \tilde{\eta}} - \tfrac{7}{8}\tilde{\eta}\frac{\partial^2 \tilde{p}_1}{\partial \tilde{\eta}^2} + 2\tilde{u}_0\frac{\partial \tilde{p}_1}{\partial \tilde{\eta}} + \tilde{v}_0\frac{\partial^2 \tilde{p}_1}{\partial \tilde{\eta}^2} = 0. \tag{3.2c}$$

The solution of these, essentially inviscid, equations results in a velocity of slip at the boundary that must be corrected in an inner viscous layer. To determine this we write, for $\tilde{\eta} \ll 1$,

$$\tilde{u}_0 = \alpha_0 + \dots, \quad \tilde{v}_0 = -2\alpha_0\tilde{\eta} + \dots, \quad \tilde{p}_1 = \beta_0 + \beta_1\tilde{\eta} + \dots. \tag{3.3}$$

Substituting into (3.2b, c), and setting $\tilde{\eta} = 0$, gives

$$\alpha_0 + \alpha_0^2 + 2\beta_0 = 0, \quad 16\alpha_0 + 9 = 0,$$

from which we have

$$\alpha_0 = -\tfrac{9}{16}, \quad \beta_0 = \tfrac{63}{512}. \tag{3.4a, b}$$

The viscous terms of (2.6) will be restored in a region of thickness $O(\tau^{1/2})$. Since $\theta_1 = O(1)$ in this region variations of p_1 across it are, from (2.6c), $O(\tau^{1/2})$. This implies that the induced pressure gradient is uniform, and $O(\tau^{-2})$, in the inner boundary layer. Thus we write

$$u_0 = \tau^{-1}\bar{u}_0, \quad p_1 = \beta_0\tau^{-2}, \quad v_0 = \tau^{-1/2}\bar{v}_0, \quad \bar{y} = \tau^{1/2}\bar{\eta} \tag{3.5}$$

so that at leading order equations (2.6a, b) give

$$\frac{\partial \bar{v}_0}{\partial \bar{\eta}} + 2\bar{u}_0 = 0, \quad \bar{u}_0 + \tfrac{1}{2}\bar{\eta}\frac{\partial \bar{u}_0}{\partial \bar{\eta}} + \bar{u}_0^2 + \bar{v}_0\frac{\partial \bar{u}_0}{\partial \bar{\eta}} = \alpha_0 + \alpha_0^2 + \frac{\partial^2 \bar{u}_0}{\partial \bar{\eta}^2}, \tag{3.6}$$

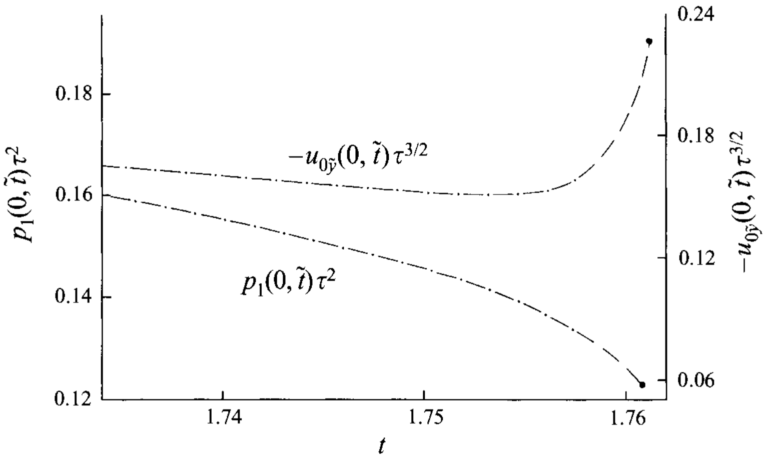

FIGURE 1. With $\tau = \tilde{t}_s - \tilde{t}$ the quantities $u_{0\tilde{y}}(0,\tilde{t})\tau^{3/2}$ ($\tilde{t}_s = 1.76130$), $p_1(0,\tilde{t})\tau^2$ ($\tilde{t}_s = 1.76084$) are shown for $b_1 < 0$, $b_0 \equiv 0$. The solution calculated from (2.6) is shown (–·–·–) with the limiting values (●) from (3.4*b*), (3.5) and (3.8); (– – –) denotes an extrapolation.

which are to be solved subject to

$$\bar{u}_0 = \bar{v}_0 = 0, \quad \bar{\eta} = 0; \quad \bar{u}_0 \to \alpha_0 \quad \text{as} \quad \bar{\eta} \to \infty. \tag{3.7}$$

The numerical solution of (3.6), subject to (3.7), yields

$$\left.\frac{\partial \bar{u}_0}{\partial \bar{\eta}}\right|_{\bar{\eta}=0} = \tau^{3/2}\left.\frac{\partial u_0}{\partial \tilde{y}}\right|_{\tilde{y}=0} = -0.2266. \tag{3.8}$$

In figure 1 we plot $-\tau^{3/2}\partial u_0/\partial \tilde{y}|_{\tilde{y}=0}$ as a function of $\tilde{t}$ with its limiting value (3.8), and also $p_1(0,\tilde{t})\tau^2$ with its limiting value of $\beta_0 = 0.12305$. These limiting forms are clearly consistent with our numerical solutions of (2.6), given the uncertainty of the precise value of $\tilde{t}_s$, adding confidence to our proposed solution structure.

In the above discussion $b_0 \equiv 0$, $b_1 < 0$. If $b_0 \neq 0$, and negative, the direct buoyancy force will oppose the self-induced pressure gradient. Indeed if $|b_0|$ is sufficiently large in that case, it will overwhelm the induced pressure gradient. There will be no eruption of the boundary layer and a steady state will be established. Similarly, if $b_0 > 0$ and $b_1 = 0$ we know from the work of Brown & Simpson (1982) that an eruption occurs. If b_1 is now increased through positive values, a value will be reached in which the self-induced pressure gradient overcomes the direct buoyancy force to yield, again, a steady flow directed away from the stagnation point. To find the critical values, more precisely the critical value of the parameter λ, say λ_c, in (2.7) we proceed as follows. For $b_1 < 0$, so that $\operatorname{sgn} b_1 = -1$, we choose a value of λ which is sufficiently large and negative that the steady-state form of equations (2.6) yields a solution. We then gradually increase the value of λ, eventually in increments $O(10^{-5})$ until the steady-state equations no longer yield a solution. For each new value of λ the initial estimate required for the solution of the nonlinear steady-state equations is taken as the solution for the previous value of λ. For $b_1 > 0, \operatorname{sgn} b_1 = 1$, we have a steady-state solution for $\lambda = 0$. Proceeding as before, increasing λ by sufficiently small values, a value is reached beyond which no steady-state solution is available. The two critical values so determined are -1.88 and 1.15 respectively. Consequently the corresponding critical relationships between b_0 and b_1 are given by

$$b_1 < 0, \quad b_0 = -1.88|b_1|^{4/5}; \quad b_1 > 0, \quad b_0 = 1.15|b_1|^{4/5}. \tag{3.9a, b}$$

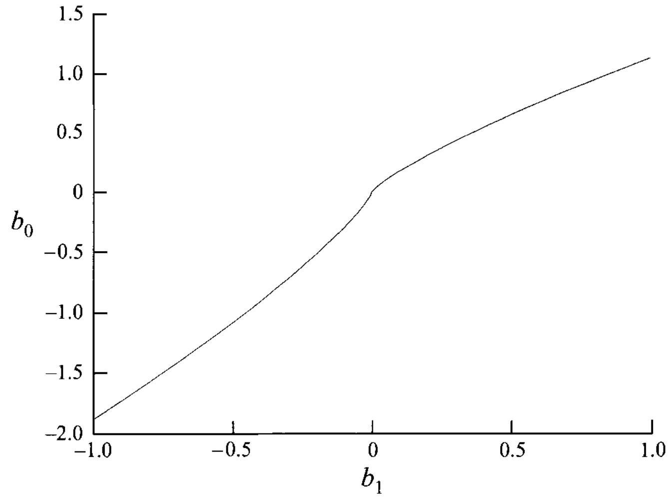

FIGURE 2. The variation of the critical value of b_0 with b_1 from (3.9). In the region below the critical value a steady flow is possible; above it the flow can never achieve a steady state.

These results are shown in figure 2. Of course, the point $b_0 = b_1 = 0$ is a singular point, and the result (3.9*a*, *b*) cannot include it.

4. Conclusions

There are essentially two mechanisms by which free-convection flows arise. The first, and most obvious, is the direct mechanism of buoyancy when a heated surface is parallel, or inclined at some angle not close to 90°, to gravity. The second is less obvious, and will be present on a differentially heated surface, even when that surface is perpendicular to gravity. In that case the free-convective effect is indirect, and relies upon the establishment of a pressure gradient parallel to the surface. The interplay between this and forced convective flows has been considered by Daniels (1992) and Amin & Riley (1995). The present paper is devoted to the rather more subtle interplay between the direct and indirect free-convective mechanisms on the near-horizontal surface at the upper stagnation point of a heated sphere. From the analysis herein we conclude that the two mechanisms may reinforce one another, either leading to an eruption of fluid from the boundary layer after a finite time, or a steady state. When in competition an eruption or steady state may again occur, depending on the relative strengths of the two mechanisms.

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